

Spherical spaces for cosmic topology and multipole selection rules.

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November 19, 2010

Abstract.

Spherical manifolds yield cosmic spaces with positive curvature. They result by closing pieces from the sphere used by Einstein for his initial cosmology. Harmonic analysis on the manifolds aims at explaining the observed low amplitudes at small multipole orders of the cosmic microwave background. We analyze assumptions of point symmetry and randomness for spherical spaces. There emerge four spaces named orbifolds, with low volume fraction from the sphere and sharp multipole selection rules in their eigenmodes.

1 Introduction.

Einstein in his initial cosmology [3], see von Laue [14] pp. 160-4 and Misner et al. [16] pp. 704-10, links gravitation by his field equations to the Riemannian metric. In his approach, the mass distribution in the cosmos into stars, galaxies and their cluster is smoothed out into a mass fluid. He closes the cosmic 3-space into a finite sphere of dimension three, for short the 3-sphere S^3 .

The motivation for considering other spherical cosmic spaces comes from observed multipole properties of the cosmic microwave background CMB radiation: Low amplitudes found for the lowest multipole orders suggest selection rules from the structure of cosmic 3-space. If cosmic 3-space is closed on a fraction of the circumference of Einstein's 3-sphere, any eigenmodes living on it must have shorter intervals of repetition, and must form a selected subset from all general eigenmodes of the 3-sphere.

All spherical 3-manifolds \mathcal{M} are variants of Einstein's cosmos and share its positive curvature. They are pieces from the 3-sphere, but are closed by a homotopic identification of their boundaries. Einstein's 3-sphere is simply connected. In contrast, spherical topologies provide multiple connections and give various fundamental groups [4], [11], see Table 4.

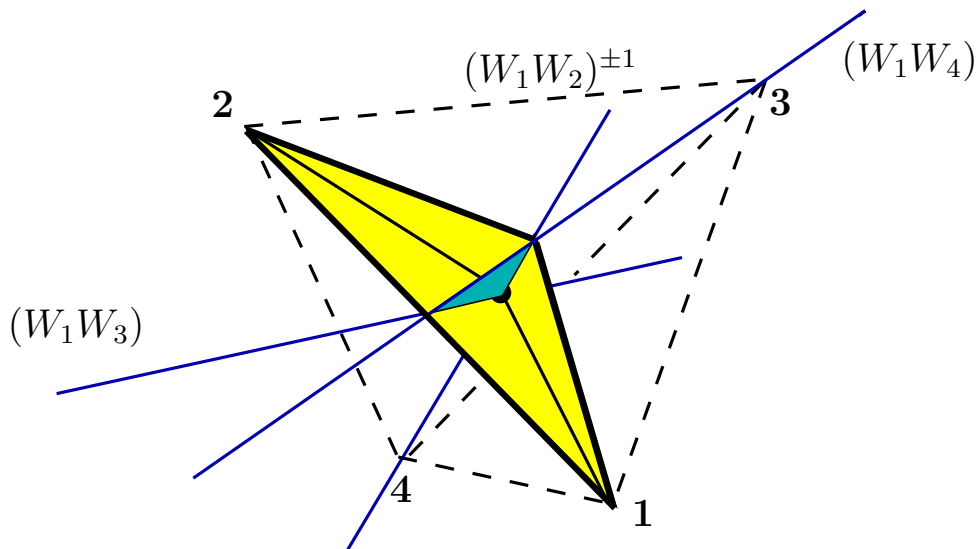


Figure 1: **Tetrahedral orbifold.** Face gluing deck rotations of the tetrahedral duplex orbifold (yellow) $N8$ in Euclidean version. The vertices of the Platonic tetrahedron are marked with numbers $(1, 2, 3, 4)$. The four covering rotations (W_1W_3) , $(W_1W_2)^{\pm 1}$, (W_1W_4) of the orbifold as products of Weyl reflections from $\Gamma = \circ - \circ - \circ - \circ$ are marked by rotations axes, intersection lines of Weyl planes. The three intersections of these lines with the orbifold form the glue triangle (light blue area). Its inner edge points have the orders $(2, 3, 2)$ of their covering rotations.

Cosmic topology looks at CMB radiation on particular manifolds as candidates for cosmic 3-space. Random CMB amplitudes from the surface of last scattering are simulated with manifold-specific bases and multipole selection rules. To achieve these goals for Platonic manifolds, we present here new tools needed from topology, group theory and functional analysis. We argue with [11], [12], [13] that proper random functions on a spherical manifold should be independent of point-symmetric positions. In terms of topology this leads from manifolds to orbifolds. We study their novel deck groups in section 5, functional analysis and multipole selection rules in section 6, boundary conditions in section 7, and bases in section 8. In Appendix A we illuminate the concepts of orbifolds and random point symmetry on the square/torus example. In Appendix B we give the representation for products of Weyl reflections.

2 Spherical spaces and eigenmodes.

Spherical spaces as topological manifolds are abstractly described as the quotients of the covering manifold S^3 and the group H of order $|H|$ acting fixpoint-free. The topologies are then classified as space forms $\mathcal{M} = S^3/H$. They were characterized with faithful

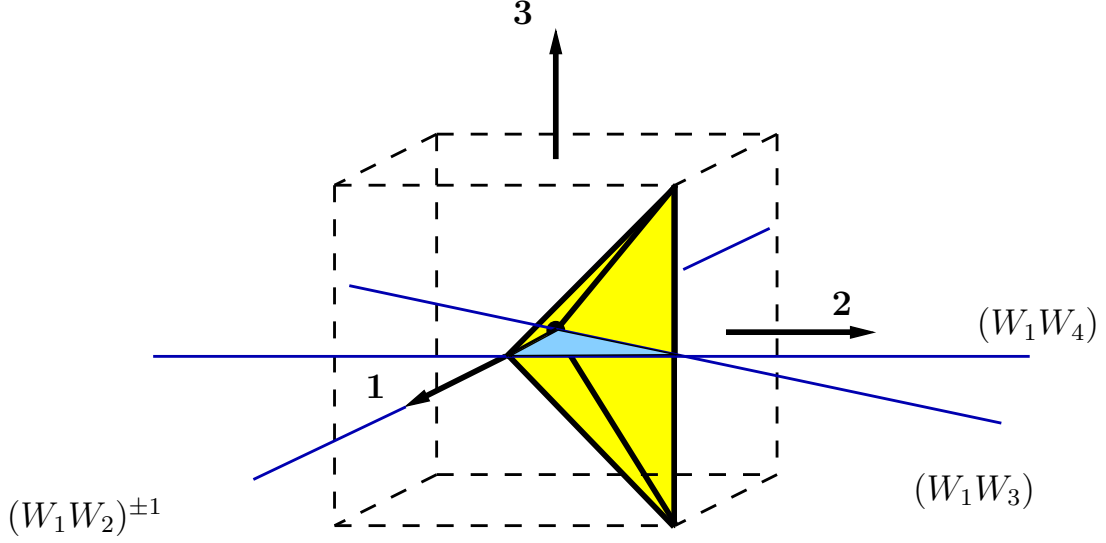


Figure 2: **Cubic orbifold.** Face gluing deck rotations of the cubic duplex orbifold (yellow) $N9$ in Euclidean version. Its four covering rotations (W_1W_3) , $(W_1W_2)^{\pm 1}$, (W_1W_4) are products of two Weyl reflections from $\Gamma = \circ - \overset{4}{\circ} - \circ - \circ - \circ$ and marked by rotation axes, intersection lines of their Weyl planes. The three intersections of these lines with the orbifold form the glue triangle (light blue area). Its inner edge points have the orders (2, 4, 2) of their covering rotations.

representations of groups H by Wolf in [21] pp. 198-230. This characterization leaves open the geometry of the manifold, and so gives no access to point symmetry. Therefore we prefer a geometric approach [11], starting from homotopy with Everitt [4].

Each spherical manifold carries a specific harmonic basis of eigenmodes, invariant under H and obeying homotopic boundary conditions. Harmonic polynomial bases vanish under the Laplacian on the Euclidean space E^4 that embeds the 3-sphere.

The eigenmodes of a spherical manifold offer two alternative views on different domains:

- (i) On a single closed manifold \mathcal{M} , they allow to expand square integrable observables. These obey homotopic boundary conditions for functional values on faces and edges.
- (ii) The covering 3-sphere is tiled by copies of \mathcal{M} . Any eigenmode now must on each tile repeat its value, and fulfill the homotopic boundary conditions. It follows that the eigenmodes must display selection rules in comparison to a general polynomial basis on the 3-sphere. This second view allows for a comparison of observables for different topologies on the same domain, Einstein's 3-sphere.

In topology, these two views present (i) the local concept of homotopy of \mathcal{M} , and (ii) the concept of deck transformations from the group $H = \text{deck}(\mathcal{M})$ that generate the tiling on the 3-sphere as universal cover. Seifert and Threlfall [19] pp. 195-8 prove the equivalence of the two views: the fundamental or first homotopy group $\pi_1(\mathcal{M})$ is isomorphic to the group of deck transformations $\text{deck}(\mathcal{M})$ that generates the tiling on the cover.

3 Spherical Platonic 3-manifolds.

We study here the family of Platonic spherical 3-manifolds. For each Platonic polyhedron we construct in [11] on the universal cover, the 3-sphere, a unique group H of fixpoint-free deck transformations acting on the 3-sphere as a subgroup of a Coxeter group Γ . By the theorem from [19], the deck groups H are isomorphic to, and were derived from, the fundamental or first homotopy groups constructed by Everitt [4].

Coxeter groups Γ [5] will become a main tool of the following analysis. They act on Euclidean 4-space with coordinates $x = (x_0, x_1, x_2, x_3)$, and on the 3-sphere, by four involutive Weyl reflections (W_1, W_2, W_3, W_4) in hyperplanes, see Table 4. The Coxeter diagram encodes the relations between the Weyl reflections, associated to its four nodes. A connecting line of two nodes implies $(W_i W_{i+1})^3 = e$, a connecting line with integer superscript $k > 3$ implies $(W_i W_{i+1})^k = e$, Weyl reflections for nodes not connected by lines commute with one another. Table 4 reviews the relation of the Platonic polyhedra to Coxeter groups. In Table 5 we list the four unit vectors $a_j \in E^4$, perpendicular to the Weyl reflection hyperplanes, for each Coxeter group. The Coxeter group tiles the 3-sphere into $|\Gamma|$ Coxeter simplices. The initial polyhedron consists of those Coxeter simplices which share a vertex at $x = (1, 0, 0, 0)$. In topology we prefer orientable manifolds. In the Coxeter groups this means that we restrict attention to the subgroups generated by an even number of Weyl reflections. We call the corresponding subgroups $S\Gamma$, where S stands for unimodularity of the defining representation on the Euclidean space E^4 . The order of these subgroups is $|S\Gamma| = |\Gamma|/2$. A set of generators of all unimodular Coxeter groups with four Weyl reflections is given by

$$S\Gamma : \{(W_1 W_2), (W_2 W_3), (W_3 W_4)\}. \quad (1)$$

Note that any product $(W_i W_j)$ leaves invariant the intersection of the two Weyl reflection hyperplanes perpendicular to the vectors $\{a_i, a_j\}$. The representation of the product is given in Appendix B,

4 Duplices under the point group tile a polyhedron.

In [11] we in addition introduce for random functions the notion of random symmetry under the point group M of these manifolds. We argue there that the values of a proper random function on a polyhedral manifold with point symmetry should be independent of operations from M . Therefore one should explore M for the analysis of the CMB radiation. The point groups are the tetrahedral group $A(4)$ for the tetrahedral 3-manifold, the cubic group O for the cubic and octahedral 3-manifolds, and the icosahedral group \mathcal{I} for the spherical dodecahedron. These point groups are unimodular subgroups w.r.t the 2-sphere

$$x_1^2 + x_2^2 + x_3^2 = 1. \quad (2)$$

and may be characterized on 3-space by unimodularity S and Coxeter subdiagrams as

$$A(4) = S(\circ - \circ - \circ), \quad O = S(\circ \overset{4}{-} \circ - \circ) \sim S(\circ - \circ \overset{4}{-} \circ), \quad \mathcal{I} = S(\circ - \circ \overset{5}{-} \circ). \quad (3)$$

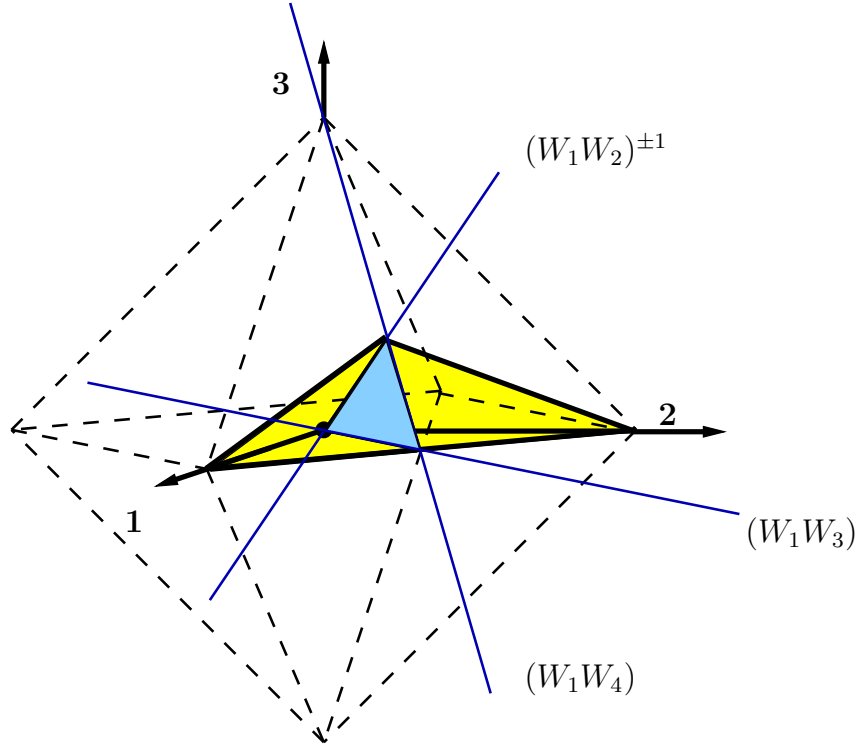


Figure 3: **Octahedral orbifold.** Face gluing deck rotations of the octahedral duplex orbifold (yellow) $N10$ in Euclidean version. The corresponding four products $(W_1W_3), (W_1W_2)^{\pm 1}, (W_1W_4)$ of two Weyl reflections from $\Gamma = \circ - \circ \overset{4}{-} \circ - \circ$ are marked by rotation axes, intersection lines of their Weyl planes. The three intersections of these lines with the orbifold form the glue triangle (light blue area). Its inner edge points have the orders $(2, 3, 2)$ of their covering rotations.

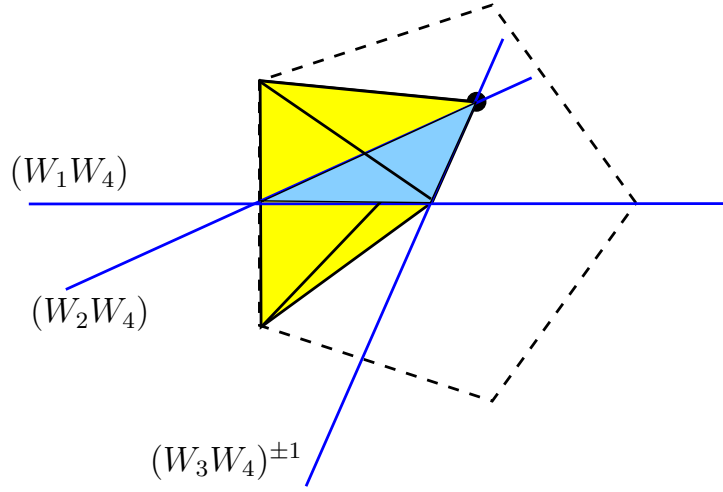


Figure 4: **Dodecahedral orbifold.** Face gluing deck rotations of the dodecahedral duplex orbifold (yellow) $N11$, attached to a dodecahedral face, in Euclidean version. The corresponding four covering products (W_2W_4) , $(W_3W_4)^{\pm 1}$, (W_1W_4) of two Weyl reflections from $\Gamma = \circ - \circ - \circ - \overset{5}{\circ}$ are marked by the intersection lines of their Weyl planes. The three intersections of these lines with the orbifold form the glue triangle (light blue area). Its inner edge points have the orders $(2, 5, 2)$ of their covering rotations.

The action of the point group M on the Platonic proto-polyhedron can now be decomposed into a fundamental domain and its images under M . The shape of the fundamental domain is not unique, but we can choose it in compact, convex and polyhedral form. Each fundamental domain for M we take as a duplex inside the proto-polyhedron, formed by gluing a Coxeter simplex and its mirror image under reflection in one simplex face.

Prop 1: Fundamental domains under point groups: For each Platonic polyhedron, a duplex fundamental domain of their point group M may be chosen as shown in Figures 1 - 4. All of them are again simplices with four faces. Images under M of this duplex tile the polyhedron.

5 Orbifolds under ST tile the 3-sphere.

The Platonic 3-manifolds under their deck group H in turn tile the 3-sphere into $|H|$ copies of the Platonic polyhedron as prototile. These tilings into Platonic polyhedra are the m -cells, $m = |H|$, described in [20] and [11]. By composing the polyhedra, tiled into duplexes, as tiles of the $|H|$ -cell tilings, it follows that the 3-sphere is tiled into $|ST| = |M| \cdot |H|$ duplexes. With respect to actions on the 3-sphere, the Coxeter duplexes, which originated as fundamental domains of the point group acting on the Platonic prototile, now become the fundamental domains of the much bigger groups ST . We shall identify them as orbifolds.

5.1 The action of deck and point groups.

The subgroup $H < ST$ of deck transformations for a given 3-manifold acts on the 3-sphere without fixpoints. The point group $M < ST$ by definition preserves the center of the polyhedron. It follows [13] that the intersection of these two subgroups consists of the identity, $H \cap M = e$.

Any image of the orbifold as proto-duplex under ST on the 3-sphere has a unique compound address, composed of a unique point group element p acting on the initial polyhedron, followed by a unique deck transformation h from the initial polyhedron to an image on the $|H|$ -cell tiling. This leads to the following general conclusion on the group structure:

Prop 2: Unimodular Coxeter groups are products of subgroups : Given a prototile duplex, its image tile in the Coxeter duplex tiling of the 3-sphere results from the action of a group element from ST , uniquely factorized into a deck and a point transformation from $H \cdot M$. This means that the elements $g \in ST$ obey the unique subgroup product law

$$g \in ST : g = hp, h \in H, p \in M, H \cap M = e, |ST| = |H| \cdot |M|. \quad (4)$$

Proof: The group ST with subgroups H, M all have faithful (one to one) representations, [21] p. 138, on E^4 . This implies that any image under $g \in ST$ of the orbifold as proto-duplex has a unique compound address as a product $g = hp, h \in H, p \in M$. Similarly one can construct for the same image a unique compound address $g = p'h'$.

This product structure $ST = H \cdot M$ differs from a direct or semidirect product. The two subgroups H, M do not commute, and the group H is not invariant under conjugation with elements from M , except in case of the cubic Coxeter group, analyzed in [11]. Eq. 4 shows that each of the subgroups generates the cosets for its partner.

5.2 Topology and deck groups for orbifolds.

We turn to the topological significance of the fundamental duplex domains for the group ST . A duplex of a unimodular Coxeter group on the 3-sphere cannot form a topological 3-manifold, because under the point group action it exhibits fixpoints of finite order $k > 1$ on its boundaries. The order k of a point is defined as the order of its stabilizer. Points of order $k = 1$ in topology are called regular, for order $k > 1$ singular, [18] pp. 664-6. We propose here to move in topology from the standard space forms to orbifolds. This concept is illustrated on the square and torus in Appendix A.

We refer to Montesinos [17] pp. 78-97 and to Ratcliffe [18] pp. 652-714 for the introduction and mathematical terms associated with orbifolds. The notion includes a manifold structure and a covering, but admits singular points of finite order.

We claim:

Prop 3: Spherical Platonic 3-manifolds with point symmetry are appropriately described by orbifolds.

We shall demonstrate orbifold coverings by deck transformations from ST , and derive the harmonic analysis for their use in cosmic topology.

In figures 1, 2, 3, 4 we reproduce, with minor changes, based on [12] a set of fundamental duplex domains for the point groups. Now we interpret them as 3-orbifolds under ST acting on the 3-sphere, based on the tetrahedron, the cube, the octahedron or the dodecahedron, drawn in their Euclidean version. The edges of the Platonic polyhedra are given in dashed lines, the orbifolds are marked by yellow color.

For topological 3-manifolds, the deck group on the 3-sphere is generated by the operations which map the pre-image of the manifold to all its face neighbours. We present a corresponding analysis for the deck group of an orbifold on the 3-sphere.

We use the tetrahedral manifold $N2$ from Table 4 for demonstration of the covering. Here the Coxeter group is $S(5)$, the symmetric group on 5 objects, with unimodular subgroup $ST = A(5)$, the group of even permutations. In Fig. 1 we mark the four vertices of the tetrahedron by the numbers 1, 2, 3, 4. A different enumeration is used in [11], Fig. 4. The Weyl reflection operators are in one-to-one correspondence to the transpositions

$$W_1 = (1, 2), W_2 = (2, 3), W_3 = (3, 4), W_4 = (4, 5). \quad (5)$$

From eq. 3 we find for the four covering deck rotations of this orbifold the simple expressions

$$\begin{aligned} (W_1 W_3) &= (1, 2)(3, 4), \\ (W_1 W_2) &= (1, 2)(2, 3) = (1, 2, 3), (W_1 W_2)^{-1} = (2, 3)(1, 2) = (3, 2, 1), \\ (W_1 W_4) &= (1, 2)(4, 5), \end{aligned} \quad (6)$$

compare Fig. 1. These even permutations generate $A(5)$.

We return to all four orbifolds associated with Platonic polyhedra and denote them by $N8, N9, N10$ as in [12] and by $N11$ for the Platonic dodecahedron. Each face of the orbifold as proto-tile of the duplex tiling is covered by a face-neighbour, a copy of the orbifold. Instead of drawing the neighbours we give in the figures the deck rotations from ST which map the preimage into its face neighbours. These deck rotations have axes which intersect with faces of the orbifold. In the figures we denote the rotation axes by blue lines and give in each case the even product of Weyl reflections. The first three deck rotations all have axes passing through the center of the polyhedron and generate the point group M . The last even product always involves a Weyl reflection that passes through an outer face of the Platonic polyhedron. The rotation containing this Weyl reflection has order $k = 2$. It transforms the orbifold into a face neighbour inside a new Platonic neighbour polyhedron. By examining the deck operations and comparing with eq. 1, it can be verified in each case that the four deck operations that cover the faces of the orbifold generate the full unimodular Coxeter group ST . This is in full analogy to the role of deck groups H for spherical 3-manifolds [11].

The intersections of the covering rotation axes with the 3-orbifold determine singular points and their order. The center point of the initial spherical proto-polyhedron, chosen as $x = (1, 0, 0, 0)$, has the maximal order $k = |M|$. For the orbifolds under inspection, more fixpoints appear on the inner points of edges of the glue triangle of each duplex, with area marked in the figures in light blue. The order k of these fixpoints agrees with the order of

Coxeter diagram Γ	$ ST = H \cdot M $	polyhedron, orbifold	$M, M $	deck generators of orbifold	order k
$\circ - \circ - \circ - \circ$	$5 \cdot 12$	tetrah., N8	$A(4), 12$	$(W_1W_3), (W_1W_2)^{\pm 1}, (W_1W_4)$	$(2, 3, 2)$
$\overset{4}{\circ} - \circ - \circ - \circ$	$8 \cdot 24$	cube, N9	$O, 24$	$(W_1W_3), (W_1W_2)^{\pm 1}, (W_1W_4)$	$(2, 4, 2)$
$\circ - \overset{4}{\circ} - \circ - \circ$	$24 \cdot 24$	octah., N10	$O, 24$	$(W_1W_3), (W_1W_2)^{\pm 1}, (W_1W_4)$	$(2, 3, 2)$
$\circ - \circ - \overset{5}{\circ} - \circ$	$120 \cdot 60$	dodecah., N11	$\mathcal{I}, 60$	$(W_2W_4), (W_3W_4)^{\pm 1}, (W_1W_4)$	$(2, 5, 2)$

Table 1: 4 Coxeter groups Γ , $|ST|$, Platonic polyhedra, orbifolds, point groups M , deck generators of orbifolds, and selected orders k for inner points on edges of the glue triangle of four duplex orbifolds $N8, N9, N10, N11$. $A(4)$ is the tetrahedral, O the cubic, \mathcal{I} the icosahedral rotation group. The center of the polyhedron has order $|M|$, the inner points of the faces of the orbifolds have order $k = 2$, inner points of the orbifolds are regular. Note that the Weyl reflections W_i depend on the Coxeter group chosen, see Table 5.

the covering rotation. Note that, in contrast to the Euclidean drawings, we always infer the order k of the rotations from the Coxeter group relations and their spherical settings on the 3-sphere. We give the covering rotations and orders of these fixpoints in Table 1. The obvious rotation axes from the point group M determine additional singular points on inner positions of edges of the orbifolds. For all four orbifolds we find:

Prop 4: Deck transformations of orbifolds generate unimodular Coxeter groups: Deck rotations cover the four faces of the four Platonic 3-orbifolds, they generate the duplex tiling of S^3 by all elements of the corresponding unimodular Coxeter deck groups ST . It follows that the orbifolds can abstractly be characterized as quotient spaces S^3/ST .

6 Harmonic analysis on orbifolds.

The harmonic analysis can follow the two views described in section 2. Clearly the homotopic boundary conditions imply selection rules compared to the full basis on the 3-sphere. The CMB radiation on an assumed topology usually is modelled by random coefficients in the specific basis. Here we analyze the basis construction, but postpone any numerical modelling.

On the 3-sphere, after replacing the coordinates x by matrix coordinates $u = u(x) \in SU(2, C)$ [11], see Appendix B, the harmonic basis may be spanned by Wigner polynomials $D_{m_1 m_2}^j(u)$ [2] or by spherical polynomials $\psi(jlm)(u)$, with linear relations

$$\begin{aligned} \psi(jlm)(u) &= \sum_{m_1 m_2} D_{m_1 m_2}^j(u) \langle j - m_1 j m_2 | lm \rangle (-1)^{(j-m_1)}, \\ D_{m_1 m_2}^j(u) &= \sum_{lm} \psi(jlm)(u) \langle j - m_1 j m_2 | lm \rangle (-1)^{(j-m_1)}, \quad 0 \leq l \leq 2j, \end{aligned} \tag{7}$$

and with summations restricted by the Wigner coefficients [2]. The spherical polynomials

transform under rotations of 3-space (x_1, x_2, x_3) like the spherical harmonics Y_m^l [11], used in the CMB data analysis. Point symmetry selects the lowest multipole order, compare [11] Table 3, [12]. The Wigner polynomial basis is suited for the projection of the subbasis invariant under the deck group H . In the cubic case, the deck group of the 3-manifold $N3$ is the quaternion group $H = Q$. Invariant under conjugation with the cubic point group O , it forms a semidirect product $ST = (Q \times_s O)$.

The harmonic analysis on the four spherical orbifolds $N8, N9, N10, N11$ is given by polynomials invariant under their unimodular Coxeter group ST . These and only these polynomials repeat their values on any Coxeter duplex from the tiling.

Prop 5: Harmonic analysis on spherical 3-orbifolds: The harmonic analysis on a spherical orbifold is spanned by harmonic polynomials, invariant under its group of deck transformations ST .

We have seen in Prop. 2 that any element of ST admits a unique factorization. For the projectors to the identity representation of ST we claim by use of eq. 2:

Prop 6: Factorization of projectors: The projector to the identity representation, denoted by Γ_1 , for the unimodular Coxeter group ST factorizes into the product of the projectors to the identity representations Γ_1 for the two subgroups H, M ,

$$P_{ST}^{\Gamma_1} = P_H^{\Gamma_1} P_M^{\Gamma_1} = P_M^{\Gamma_1} P_H^{\Gamma_1}. \quad (8)$$

Proof: In the group operator algebra of the unimodular Coxeter group ST we have from eq. 3 the factorization

$$\begin{aligned} P_{ST}^{\Gamma_1} &= \sum_g T_g = \sum_{h,p} T_{hp} = \sum_{p,h} T_{ph}, \quad h \in H, p \in M, \\ &= \sum_{h,p} T_h T_p = \left(\sum_h T_h \right) \left(\sum_p T_p \right) = \left(\sum_p T_p \right) \left(\sum_h T_h \right) = P_H^{\Gamma_1} P_M^{\Gamma_1} = P_M^{\Gamma_1} P_H^{\Gamma_1}. \end{aligned} \quad (9)$$

This result greatly simplifies the projection to the identity representation:

The projectors $P_H^{\Gamma_0}$ for the group H of deck transformations are given in [11] in the Wigner polynomial basis, see eq. 12. So it remains to pass with eq. 7 from the Wigner to the spherical basis, and then to apply the projectors $P_M^{\Gamma_0}$ of the point group.

Prop 7: Orbifolds give sharp multipole selection rules: ST -invariant polynomials by their point group M -invariance select a lowest non-zero multipole order l , see [11] Table 3. This projection is carried out in [12] for the spherical cubic 3-orbifold and multipole order $0 \leq l \leq 8$. The results are reproduced here in Tables 2 and 3 in explicit multipole order. The analysis can be extended by the method described in section 8.

A similar analysis applies to the spherical tetrahedral, octahedral and dodecahedral 3-orbifolds.

7 Homotopic boundary conditions from orbifolds.

From [11] we know that a fixed geometric shape of a Platonic 3-manifold can have different and inequivalent topologies, characterized by different groups of homotopies and deck

l	$Y^{\Gamma_1, l} = \sum_m a_{lm} Y_m^l(\theta, \phi)$
0	Y_0^0
4	$\sqrt{\frac{7}{12}} Y_0^4 + \sqrt{\frac{5}{24}} (Y_4^4 + Y_{-4}^4)$
6	$\sqrt{\frac{1}{72}} Y_0^6 - \sqrt{\frac{7}{144}} (Y_4^6 + Y_{-4}^6)$
8	$\frac{1}{64} \sqrt{33} Y_0^8 + \frac{1}{12} \sqrt{\frac{21}{2}} (Y_4^8 + Y_{-4}^8) + \frac{1}{24} \sqrt{\frac{195}{2}} (Y_8^8 + Y_{-8}^8)$

Table 2: The lowest cubic O -invariant spherical harmonics $Y^{\Gamma_1, l}$, expressed by spherical harmonics Y_m^l .

$2j$	l	$\psi^{0, \Gamma_1, 2j} = \sum_l b_l R_{2j+1, l}(\chi) Y^{\Gamma_1, l}(\theta, \phi)$
0	0	$R_{10} Y^{\Gamma_1, 0}$
4	0, 4	$\sqrt{\frac{2}{5}} R_{50} Y^{\Gamma_1, 0} + \sqrt{\frac{3}{5}} R_{54} Y^{\Gamma_1, 4}$
6	0, 4, 6	$\sqrt{\frac{1}{7}} R_{70} Y^{\Gamma_1, 0} - \sqrt{\frac{6}{11}} R_{74} Y^{\Gamma_1, 4} - \sqrt{\frac{24}{77}} R_{76} Y^{\Gamma_1, 6}$
8	0, 4, 6, 8	$\frac{4}{3} \sqrt{\frac{1}{110}} R_{90} Y^{\Gamma_1, 0} - \frac{12}{11} \sqrt{\frac{3}{65}} R_{94} Y^{\Gamma_1, 4}$ $+ \frac{8 \cdot 19}{165} R_{96} Y^{\Gamma_1, 6} + \frac{4}{5} \sqrt{\frac{1}{33 \cdot 13}} R_{98} Y^{\Gamma_1, 8}$

Table 3: The lowest $(S\Gamma = (Q \times_s O))$ -invariant polynomials $\psi^{0, \Gamma_1, 2j}$ of degree $2j$ on the 3-sphere, expressed by the cubic invariant spherical harmonics from Table 2. $(Q \times_s O)$ -invariance enforces superpositions of several cubic invariant spherical harmonics.

transformations H . These differences give rise to different homotopic boundary conditions. We now examine the boundary conditions for orbifolds.

We have seen that the orbifold is covered face-to-face by rotational images. It follows, as in the case of 3-manifolds, that the topology on 3-manifolds implies homotopic boundary conditions on the faces of the 3-orbifolds. Since with the orbifolds we introduce point symmetry in addition to deck transformations, we find from the arguments given in [11] for point symmetry,

Prop 8: Topological universality from point symmetry: If we demand, for a function on a given Platonic 3-manifold, point symmetry under M in addition to the boundary conditions set by homotopy on faces and edges, then new boundary conditions apply universally, that is, independent of the specific group of deck transformations chosen.

Prop 9: Universal homotopic boundary conditions from 3-orbifolds: If in addition to homotopy we demand on the manifold symmetry under the rotational point group M , the homotopic boundary conditions for different deck and homotopy groups on the same Platonic geometrical shape coincide with one another and reduce to the homotopic boundary conditions for the orbifolds. Their boundary conditions are determined by the generators in Table 1 of the covering rotations.

Coxeter diagram Γ	$ \Gamma $	polyhedron \mathcal{M}	$H = \text{deck}(\mathcal{M})$	$ H $	Reference
$\circ - \circ - \circ - \circ$	120	tetrahedron $N1$	C_5	5	[7]
$\circ - \overset{4}{\circ} - \circ - \circ$	384	cube $N2$ cube $N3$	C_8 Q	8 8	[8] [11]
$\circ - \circ - \overset{4}{\circ} - \circ$	1152	octahedron $N4$ octahedron $N5$ octahedron $N6$	$C_3 \times Q$ B \mathcal{T}^*	24 24 24	[11] [11] [11]
$\circ - \circ - \circ - \overset{5}{\circ}$	120^2	dodecahedron $N1'$	\mathcal{J}^*	120	[6]

Table 4: 4 Coxeter groups Γ , 4 Platonic polyhedra \mathcal{M} , 7 deck groups $H = \text{deck}(\mathcal{M})$ of order $|H|$. C_n denotes a cyclic, Q the quaternion, \mathcal{T}^* the binary tetrahedral, \mathcal{J}^* the binary icosahedral, $S\Gamma$ a unimodular Coxeter group. The symbols Ni are taken from [4].

Γ	a_1	a_2	a_3	a_4
$\circ - \circ - \circ - \circ$	$(0, 0, 0, 1)$	$(0, 0, \sqrt{\frac{3}{4}}, \frac{1}{2})$	$(0, \sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}, 0)$	$(\sqrt{\frac{5}{8}}, \sqrt{\frac{3}{8}}, 0, 0)$
$\circ - \overset{4}{\circ} - \circ - \circ$	$(0, 0, 0, 1)$	$(0, 0, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$	$(0, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, 0)$	$(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, 0, 0)$
$\circ - \circ - \overset{4}{\circ} - \circ$	$(0, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, 0)$	$(0, 0, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$	$(0, 0, 0, 1)$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$\circ - \circ - \circ - \overset{5}{\circ}$	$(0, 0, 1, 0)$	$(0, -\frac{\sqrt{-\tau+3}}{2}, \frac{\tau}{2}, 0)$	$(0, -\sqrt{\frac{\tau+2}{5}}, 0, -\sqrt{\frac{-\tau+3}{5}})$	$(\frac{\sqrt{2-\tau}}{2}, 0, 0, -\frac{\sqrt{\tau+2}}{2})$

Table 5: The Weyl vectors a_s for the 4 Coxeter groups Γ from Table 4 with $\tau := \frac{1+\sqrt{5}}{2}$.

8 Recursive computation of the bases invariant under the orbifold deck groups $S\Gamma = H \cdot M$.

We describe the recursive construction of $S\Gamma$ -invariant bases by use of the factorization given in Prop. 4. This recursive construction was used to obtain the results of Tables 2, 3. The multiplicity $m(l, \Gamma_1)$ of the identity representation Γ_1 of M for given multipole order l is given in [15] pp. 436-8.

Our basic tool are the relations [11] between Wigner and spherical harmonic polynomials on the 3-sphere given in eq. 7.

(i) We start from a linear combination of spherical harmonics of multipole order l , invariant under the point group M , and construct with its coefficients a_{lm} a M -invariant linear combination of spherical polynomials $\psi_{jlm}(u)$, $l \leq 2j$ eq. 7, on the 3-sphere [11],

$$\psi_{j,l,\Gamma_1}(u) = \sum_m^l a_{lm}^{\Gamma_1} \psi_{j,l,m'}(u) \quad (10)$$

The starting linear combinations of spherical harmonics for lowest multipole order l for the

point groups involved can be found in the literature. Note that the coefficients $a_{lm'}^{\Gamma_1}$ are independent of the degree $2j$.

(ii) Next we transform with eq. 7 from the spherical to the Wigner basis and apply the projector $P_H^{\Gamma_1}$ from [11] for H -invariance. If the resulting function does not vanish, we transform it back to the spherical basis. In this way we find from the starting function a new one, invariant from Prop. 4 under both M and H and hence under ST , given by

$$\begin{aligned}\psi_{j,\Gamma_1}(u) &= \sum_i \left[\sum_m a_{l+i,m}^{\Gamma_1} \psi_{j,l+i,m}(u) \right], \\ a_{l+i,m}^{\Gamma_1} &= \sum_{m',m'_1,m'_2,m_1,m_2} a_{lm'} \langle j - m'_1 j m'_2 | l m' \rangle (-1)^{(j-m'_1)} \\ &\quad \langle j m'_1 m'_2 | P_H^{\Gamma_1} | j m_1 m_2 \rangle \langle j - m_1 j m_2 | l + i m \rangle (-1)^{(j-m_1)},\end{aligned}\tag{11}$$

The polynomial eq. 11 if non-vanishing can be normalized.

The Wigner coefficients are given in [2], and the matrix elements of the projector $P_H^{\Gamma_1}$ for the identity representation of the group H are given in the Wigner polynomial basis by

$$\langle j m'_1 m'_2 | P_H^{\Gamma_1} | j m_1 m_2 \rangle = \frac{1}{|H|} \sum_{h=(h_l, h_r) \in H} D_{m'_1 m_1}^j(h_l^{-1}) D_{m'_2 m_2}^j(h_r).\tag{12}$$

and specified in [11] for each Platonic 3-manifold. Due to universality Prop. 5, we can choose the most convenient deck group H for a fixed geometric shape. The recursion relation eq. 11 involves Wigner coefficients, the elements of the group H given as pairs $h = (h_l, h_r)$ in [11], and Wigner D^j -representations for the group $SU(2, C)$. The ST -invariant bases appear as linear combinations of M -invariant spherical functions with fixed multipole order $l + i$.

(iii) Moreover, since the point group action cannot change the multipole order, each new partial sum of eq. 11 in square brackets for fixed $l + i$ must separately be invariant under the point group M . This allows to restart the computation with $l \rightarrow l + i$ by going again from spherical to Wigner polynomials, followed by projection of an invariant under H . In this way we can increase the polynomial degree $2j$. By character technique we can control the number of invariants for given degree $2j$ of the polynomials.

9 Conclusion.

We propose the interpretation and use of 3-orbifolds in cosmic topology along two lines:

(I) The notion of a topological 3-manifolds with (random) point symmetry M is reformulated in terms of topological 3-orbifolds. The relevant group of deck transformations is a unimodular Coxeter group ST . This group generates, from a Coxeter duplex orbifold as prototile, a tiling of the 3-sphere into $|ST|$ copies. The group $ST = H \cdot M$ combines the group H of polyhedral deck transformations with the point group M of the polyhedron. The Platonic 3-manifolds under the assumption of random point symmetry shrink into

3-orbifolds, and their harmonic bases live on a fraction $1/|ST|$ of the 3-sphere, see Table 1. They are more selective than those for the 3-manifolds, and therefore easier to test.

For the deck group $ST = H \cdot M$ we construct a new harmonic analysis which can model the CMB radiation. Point symmetry implies sharp multipole selection rules, and topological universality: all fundamental groups for the same geometrical polyhedral shape produce the same boundary conditions. There is only a single point-symmetric harmonic basis, invariant under both H and M . This result is demonstrated in Tables 2, 3 from [12] for orbifolds from spherical cubes.

(II) We can retain the strict original notion of topological 3-manifolds. The new harmonic basis, characterized by point symmetry, forms a subbasis of the harmonic analysis. If it fails to model the CMB fluctuations, one can augment it by the larger basis of [11] for the 3-manifold without point symmetry.

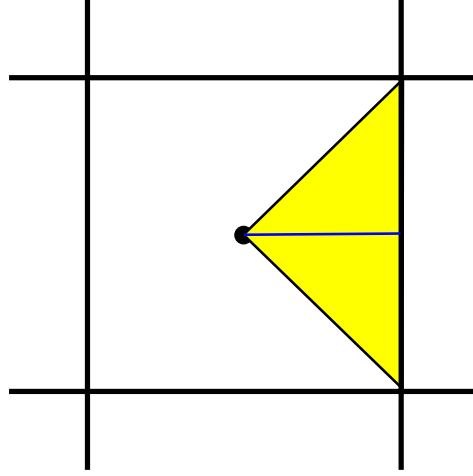


Figure 5: **From the square to its orbifold.** The square in the Euclidean plane. Parallel edges are identified by homotopy to yield the 2-torus T^2 . A $(\pi/2)$ -sector (yellow) forms a fundamental domain under the group C_4 of 4fold rotations. In topology this sector is an orbifold. The center point under C_4 has order 4.

10 Appendix A: From square and torus to orbifold.

From square to torus: The square on the Euclidean plane E^2 , see figure 5, is closed into a topological manifold of curvature zero as follows: We identify the two pairs of parallel edges - this gluing generates homotopy. The resulting topological manifold is the 2-torus T^2 , it is finite but unbounded. The 2-torus on its surface admits two types of closed loops, whose multiple windings are generated by two commuting infinite cyclic groups $C_\infty \times C_\infty$. This is the homotopy group of the 2-torus, $\pi_1(T^2) = C_\infty \times C_\infty$. The 2-torus when unfolded back into its universal cover, the plane E^2 , becomes a prototile of a square tiling. Its repetition pattern consists of two infinite translation groups in perpendicular directions. The group that generates the square tiling is the discrete two-fold translation or deck group, it is again $\text{deck}(T^2) = C_\infty \times C_\infty$. The group of homotopic windings of the 2-torus and the deck group are isomorphic and illustrate the general theorem of Seifert and Threlfall [19] pp. 195-8.

Fourier basis: Next turn to functions on the square and on the 2-torus. Complex-valued functions have the exponential basis of the twofold periodic Fourier series. The basis obeys homotopic boundary conditions: it repeats its values on parallel edges of the square tiling. This property extends to any function that can be written as a Fourier series.

Point symmetry: Now we note that the square has a point symmetry: Multiples of the rotation $R(\phi)$, $\phi = \pi/2$ map the square into itself while keeping its center. The relevant point group is the cyclic group C_4 . If we inscribe into the square a $(\pi/2)$ -sector (yellow) by connecting two endpoints of an edge to the center, we can reach any other points of the square from this sector by applying the four rotations from C_4 . The sector in topology is

called an orbifold.

Random functions: Turn in particular to random functions $f^{\text{random}}(x)$ on the square. If we want to attach them to the 2-torus as topological manifold, we must demand for functional values of $f^{\text{random}}(x)$ the homotopic boundary conditions of twofold periodicity, equivalent to allowing for an expansion into a twofold Fourier series. The Fourier coefficients may be chosen at random.

Random point symmetry: Imagine a random function $f^{\text{random}}(x)$ on the square and apply to it the rotation $R(\pi/2)$. As edges of the square are mapped into edges, the rotated function $\tilde{f}^{\text{random}}(x) := f^{\text{random}}(R^{-1}(\pi/2)x)$ is another random function on the same square. To render a proper random function independent of this point rotation, we must add the values of both functions, and, extending the argument to all four rotations, must make the random function invariant under C_4 .

Orbifold: This invariance property is achieved by shrinking the domain $\{x\}$ of definition of the random function $f^{\text{random}}(x)$ from the square to the $(\pi/2)$ -sector, that is, to the points of the orbifold introduced above. The functional values on the square follow by rotations from C_4 . Moreover the twofold Fourier basis of $f^{\text{random}}(x)$ must be restricted to its subbasis invariant under C_4 . The orbifold becomes the fundamental domain under a crystallographic space group, named the asymmetric unit.

For cosmic topology with curvature zero on a finite closed Euclidean manifold see [1].

Appendix B: Representation for products of Weyl reflections.

From [11] we recall:

The map of a unit vector x from Cartesian to $SU(2, C)$ coordinates $u(x)$ is given by

$$x = (x_0, x_1, x_2, x_3) \rightarrow u(x) = \begin{bmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{bmatrix}. \quad (13)$$

The action T_g of the rotation group $SO(4, R) \sim ((SU^l(2, C) \times SU^r(2, C))/Z_2)$ with elements $g = (g_l, g_r)$ on a Wigner polynomial is, by use of representations of $SU(2, C)$ [2],

$$(T_{(g_l, g_r)} D_{m_1, m_2}^j)(u) = D_{m_1, m_2}^j(g_l^{-1} u g_r) = \sum_{m'_1 m'_2} D_{m'_1, m'_2}^j(u) D_{m_1, m'_1}^j(g_l^{-1}) D_{m'_2, m_2}^j(g_r). \quad (14)$$

For two Weyl reflection operators with Weyl unit vectors $\{a_i, a_j\}$ define with eq. 13

$$v_i := u(a_i), \quad v_j := u(a_j). \quad (15)$$

Then the rotation operator for the product $(W_i W_j)$ in terms of eq. 14 is given by

$$T_{(W_i W_j)} = T_{(v_i v_j^{-1}, v_i^{-1} v_j)}. \quad (16)$$

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